Quiz (2) First year – Mechanical Engineering (production)

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Answer the following question:

(1) If $\vec{A} = \vec{\nabla}\phi$ where $\phi = z^2 x^3 y$ find $\vec{\nabla}\times\vec{A},\ \vec{\nabla}.\vec{A}$.

(2) Evaluate $\iint_S \vec{F}.\vec{n}\,dS$ in the case $\vec{F} = y\vec{i} + 2x\vec{j} - z\vec{k}$ and S is the plane

$2x + y = 6,\quad x, y, z \geq 0$ and cutting by $z = 4$.

(3) Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F}(t) = (2x + y)\vec{i} + (3y - x)\vec{j}$ along the curve which consists of the straight lines from $(0,0)$ to $(2,0)$ and from $(2,0)$ to $(3,2)$.

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Answer (1)

$$\overline{A} = \overline{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\overline{i} + \frac{\partial\varphi}{\partial y}\overline{j} + \frac{\partial\varphi}{\partial z}\overline{k}$$

$$= \frac{\partial}{\partial x}\left(z^2x^3y\right)\overline{i} + \frac{\partial}{\partial y}\left(z^2x^3y\right)\overline{j} + \frac{\partial}{\partial z}\left(z^2x^3y\right)\overline{k} = 3z^2x^2y\,\vec{i} + z^2x^3\overline{j} + 2z\,x^3y\,\vec{k}$$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3z^2x^2y & z^2x^3 & 2z\,x^3y \end{vmatrix} = 0$$

$$\vec{\nabla}.\vec{A} = \left(\frac{\partial}{\partial x}\overline{i} + \frac{\partial}{\partial y}\overline{j} + \frac{\partial}{\partial z}\overline{k}\right).\left(3z^2x^2y\,\vec{i} + z^2x^3\overline{j} + 2z\,x^3y\,\vec{k}\right) = 6z^2xy + 2x^3y$$

Answer: (2)

$$\varphi = 2x + y - 6,\qquad \overline{\nabla}\varphi = 2\overline{i} + \overline{j},\quad \overline{n} = \frac{2\overline{i} + \overline{j}}{\sqrt{5}}$$

$$\overline{F}.\overline{n} = \left(y\vec{i} + 2x\vec{j} - z\vec{k}\right).\left(\frac{2\overline{i} + \overline{j}}{\sqrt{5}}\right) = \frac{2y + 2x}{\sqrt{5}}$$

$$dS = \frac{dxdz}{\overline{n}.\overline{j}} = \sqrt{5}dxdz$$

$$\iint_S \vec{F}.\vec{n}\,dS = \iint_A \left(\frac{2y+2x}{\sqrt{5}}\right)\sqrt{5}\,dx\,dz = \iint_A (2y+2x)dxdz$$

Where A is the rectangle in xz-plane with vertices (3,0,0).(0,0,0),(0,0,4),(3,0,4)

$$\iint_A (2y+2x)dxdz = \iint_A [2(6-2x)+2x]dxdz = \int_0^3\int_0^4 (12-2x)dxdz = 144-36=108$$

Answer (3)

$$\therefore \int_C \vec{F}\,d\vec{r} = \int_C \left[(2x+y)\vec{i}+(3y-x)\vec{j}\right].(dx\,\bar{i}+dy\,\bar{j})$$

$$= \int_C \left[(2x+y)\,dx+(3y-x)dy\right] = \int_{(0,0)}^{(2,0)} \left[(2x+y)\,dx+(3y-x)dy\right]$$

$$+\int_{(2,0)}^{(3,2)} \left[(2x+y)\,dx+(3y-x)dy\right] = I_1+I_2 \qquad (1)$$

where $I_1 = \int_{(0,0)}^{(2,0)} \left[(2x+y)\,dx+(3y-x)dy\right]$ taken on the line

$y=0, dy=0$ and x various from 0 to 2

$$\therefore I_1 = \int_0^2 [(2x\,dx] = \left[x^2\right]_0^2 = 4 \qquad (2)$$

and $I_2 = \int_{(2,0)}^{(3,2)} \left[(2x+y)\,dx+(3y-x)dy\right]$ taken on the line joining the two points $(2,0)$,

$(3,2)$ which has an equation $\dfrac{y-0}{x-2}=\dfrac{2-0}{3-2} \Rightarrow y=2x-4 \Rightarrow dy=2dx$

$$\therefore I_2 = \int_2^3 [2x+(2x-4)]\,dx + [3(2x-4)-x]2dx$$

$$= \int_2^3 \{[4x-4]\,dx+[10x-24]\}dx = \int_2^3 [14x-28]\,dx = \left[7x^2-28x\right]_2^3 = 7$$

From (3) and (2) we have $\int_C \vec{F}.d\vec{r} = 4+7=11$